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Meromorphic quadratic differentials with prescribed singularities

Homero G. Diaz-Marin

Abstract. We study the singular flat structure associated to any meromorphic quadratic differential on a compact Riemann surface to prove an existence theorem as follows. There exists a meromorphic quadratic differential with given orders of the poles and zeros and orientability or non orientability of the horizontal foliation, iff these prescribed topological data are admissible according to the Gauss-Bonnet Theorem, the Residue Theorem and certain conditions arising from local orientability or non orientablity considerations. Some few exceptional cases remain excluded. Thus, we generalize two previous results. One due to Masur & Smillie, which assumes that poles are at most simple; and a second one due to Muciño-Raymundo, which assumes that the horizontal foliation is orientable.

Keywords: Riemann surface, singular foliation, quadratic differential, meromorphic differential.

1 Statement of the result

Consider a compact Riemann surface M, and a meromorphic quadratic differential ϕ on M having poles and/or zeros at $\Sigma = \{p_1, \dots, p_n\}$. A remarkable fact about the pair (M, ϕ) is that it provides a quite simple and natural geometric structure, namely, a flat metric in $M \setminus \Sigma$ having singularities at Σ, and whose associated holonomy group on $M \setminus \Sigma$ can be either $\{Id\}$ or $\{\pm Id\}$ (this in turn gives rise to a geodesic singular foliation called horizontal foliation).

It turns out that some other hypothesis on (M, ϕ) arise as natural conditions for certain problems. For instance, in some celebrated approaches related to

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dynamical considerations on surfaces (see **[KMS]**), one important assumption is that the total area of M, measured according to the flat metric given by ϕ, is finite. A detailed analysis of this condition, shows that all poles should be at most simple (see **[St]** for a geometrical description of the singularities of a quadratic differential).

On the other hand, if we assume, as in **[MR]**, that the holonomy group is $\{Id\}$, then we have that there exists a meromorphic Abelian differential ω such that $\omega \otimes \omega = \phi$. Moreover, in this case there exists a uniquely defined meromorphic vector field X, that satisfies $\omega(X) \equiv 1$. Thus we can establish global correspondences $\phi = \omega \otimes \omega \rightarrow \omega \leftrightarrow X$.

Note that the horizontal foliation of ϕ corresponds to the real trajectories of the associated real vector field $X + \overline{X}$. Given any ϕ, we will define $\epsilon = +1$ if its associated horizontal foliation is of the type described above, and $\epsilon = -1$ otherwise (i.e. if this foliation is non-orientable in the real sense).

We consider in this note the general case. Let ϕ be a meromorphic quadratic differential on a compact Riemann surface M, ϕ having poles of order $-k_i$ at suitable points $p_i \in \Sigma \subset M$ for $1 \leq i \leq l$, and zeros of order k_j at points $p_j \in \Sigma \subset M$ for $l + 1 \leq j \leq n$. Hence it determines certain local *topological data*, the orders of the zeros and poles, and a global one, the orientability of the horizontal foliation $\epsilon = \pm 1$. Therefore, for each pair (M, ϕ), we have the following assignment $(M, \phi) \longmapsto k := (k_1, \dots, k_l, \dots, k_n;\ \epsilon)$, where the entries k_i of k satisfy $k_i \in \mathbb{Z}^-$ for $1 \leq i \leq l$, $k_j \in \mathbb{Z}^+$ for $l + 1 \leq j \leq n$ and $\epsilon = \pm 1$ (the order of the entries is not important). In this case we say that the singular flat structure (M, ϕ) *realizes* the data k or simply that k is *realizable*.

A natural question, which will be solved by our main result, is the following

Problem. *Under which conditions any prescribed topological data* $k := (k_1, \dots, k_n;\ \epsilon)$ *are realized by a meromorphic quadratic differential* ϕ *on a Riemann surface* M.

At first sight, we can describe some obstructions for realizing any k. Let us then analyze our necessary conditions. Assume that k is realized by a singular flat structure (M, ϕ).

Consider the Gauss-Bonnet Theorem for singular flat metrics of finite area and without any holonomy assumption (see **[GKS]**), and also the Poincaré-Hopf Theorem. Both theorems relate local data about the metric or foliation singularities to a global datum, the Euler characteristic. Inspired in these two results, by translating adequately our local invariants, it is natural to propose that

k satisfies the following condition

$$\sigma(k) := \sum_{i=1}^{n} k_i = 4(g-1)$$

where g is the genus of M.

We also have some obstructions which arise from the local orientability status of the horizontal foliation associated to ϕ. First we observe that local non-orientability implies global non orientability, therefore whenever an entry k_i of k is odd, $\epsilon = -1$.

On the other hand, on the sphere, observe that the holonomy can be calculated just by making parallel transport along each loop going around each singularity on the sphere. Thus it turns out that local orientability implies global orientability of the horizontal foliation, i.e. whenever $\sigma(k) = -4$, and $k_i \in 2\mathbb{Z}$ for every i, then $\epsilon = +1$.

Furthermore, consider ϕ having a single pole of order 2 and being the square of a meromorphic differential ω. Then ω has only one simple pole, which is not possible according to the Residue Theorem for the associated meromorphic differential, (see **[MR]** Theorem 2.1). Therefore, if $k = (-2, k_2, \ldots, k_n;\ \epsilon)$ where $k_i \in 2\mathbb{Z}^+$ for $2 \leq i \leq n$, then $\epsilon = -1$.

Finally, there are some topological data that, at the first glance, have no apparent obstructions, nevertheless they can not be realizable. These data, which we call *exceptional*, are $(4; -1)$, $(1, 3; -1)$, $(-1, 1; -1)$, $(\ ; -1)$. The impossibility for each one is provided in **[MS]** by using suitable arguments of complex geometry.

Our main result is:

Theorem 1. *Let $k = (k_1, \ldots, k_n; \epsilon)$ be topological data where $k_i \in \mathbb{Z} \setminus \{0\}$ and $\epsilon \in \{-1, 1\}$. Then k is realizable by a meromorphic quadratic differential ϕ on a compact Riemann surface M, if and only if the following conditions are satisfied*

1) *$\sigma(k) = 4(g-1)$ where $0 \leq g \in \mathbb{Z}$,*

2) *whenever an entry k_i of k is odd, $\epsilon = -1$,*

3) *whenever $\sigma(k) = -4$ and $k_i \in 2\mathbb{Z}$ for every i, $\epsilon = +1$,*

4) *k is admissible according to the Residue Theorem,*

5) *k is not exceptional.*

This is just the generalization of the special cases in **[MS]** and **[MR]** so that now we are able to give a complete description of the necessary and sufficient existence conditions without adding any hypothesis on the poles or on the holonomy group. Thus, we can deal with the problem of flat singular structures yielding infinite area.

As a remaining problem, we do not know the obstructions that can arise when we consider the realization problem stiffening a complex structure on a compact surface. Since by using Theorem 1, we can not assure the existence of a flat structure that realizes certain given topological data on each conformal class.

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2 General idea of the proof

Our basic object is going to be the set $\mathcal{K}$ of all topological data that are admissible according to conditions (1) to (4) in Theorem 1. In addition, we are going to assume the following hypothesis on the elements of $\mathcal{K}$.

6) For every $k \in \mathcal{K}$, *there exists an entry* k_i *of* k, *such that* $k_i \leq -2$.

If the topological data k are such that every entry k_i of k satisfies $k_i \geq -1$, then it satisfies the conditions of the main theorem given in **[MS]**, and Theorem 1 holds for this case.

Remark 1. Assumption (6) implies that there are no exceptional data in $\mathcal{K}$. Therefore if $k \in \mathcal{K}$, then k satisfies also condition (5) of Theorem 1.

The main idea of the proof consists of two steps. First the reduction of the problem by giving a partial ordering to the set $\mathcal{K}$, in such a way that if $\kappa \in \mathcal{K}$ is realizable, then any other element k which is greater than κ (according to this ordering) is also realizable. This adequate partial ordering can be inferred from few cut-paste constructions, these operations allow us to change the order of certain singularities in the singular flat structures that realizes κ, getting one that realizes k.

Once we show the minimal elements of the partial ordering we have given to $\mathcal{K}$, we show that these minimal elements are realizable by directly giving an explicit singular flat structure for each one. In fact, these minimal elements correspond to the cases on the sphere and some other cases on the torus.

3 Trivial case on the sphere

Suppose $k = (k_1, \ldots, k_n; \epsilon) \in \mathcal{K}$ where $\sigma(k) = -4$, we have to provide the Riemann sphere $\mathbb{CP}^1$ with a metric that is flat except in a finite number of points.

Let $z_1, \ldots, z_l, \ldots, z_n$ be n distinct points in $\mathbb{C}$, then the quadratic differential

$$\frac{(z-z_{l+1})^{k_{l+1}}\cdots(z-z_n)^{k_n}}{(z-z_1)^{-k_1}\cdots(z-z_l)^{-k_l}}dz^2$$

on $\mathbb{C}$ can be extended to $\mathbb{CP}^1$, in such a way that ∞ is a regular point of ϕ. We have taken into account the condition (3) of our theorem. Obviously this quadratic differential realizes k.

4 Reduction of the problem

4.1 Partial ordering

Definition. Take $\kappa = (k_1, \ldots, k_l, k_{l+1}, \ldots, k_m; \epsilon) \in \mathcal{K}$, where $k_i \leq -2$ for $i \leq l$, and $k_j \geq -1$ for $l+1 \leq j \leq m$. We will write $\kappa \precsim k$, if k can be written as follows:

$$k = (k_1, \ldots, k_l,\ k_{l+1} + 4h_1, \ldots, k_m + 4h_{m-l},\ k_{m+1}, \ldots, k_n;\ \epsilon)$$

where h_i are non-negative integers, $i = 1, \ldots, m-l, k_j \geq -1$ for $m+1 \leq j \leq n$ and

$$\sum_{i=m+1}^{n} k_i \geq 0.$$

Remark 2. We have obtained a partial ordering $(\mathcal{K}, \precsim)$, such that every chain has a minimal element. Obviously σ is a monotone function in $(\mathcal{K}, \precsim)$, i.e., if $\kappa \precsim k$ then $\sigma(\kappa) \leq \sigma(k)$. The entries that are less or equal to -2 are the exactly same for κ and for k.

4.2 Basic constructions

Given $\kappa, k \in \mathcal{K}$ where κ is realizable and $\kappa \precsim k$, we want to conclude that k is also realizable. In order to support our claiming, in the next paragraphs we give a series of simple constructions that will allow us to reach a singular flat structure that realizes k from one that realizes κ by means of cut-paste local operations.

Construction 1. If $(k_1, \ldots, k_m;\ \epsilon) \in \mathcal{K}$ is realized by a singular flat structure (M, ϕ) and if $k_m \geq -1$, then $(k_1, \ldots, k_{m-1}, k_m + 4;\ \epsilon)$ is also realizable.

Let $q_1 \in M$ be a zero having order k_m or a simple pole (this is the case if $k_m = -1$). Consider a geodesic segment α in some critical trajectory starting

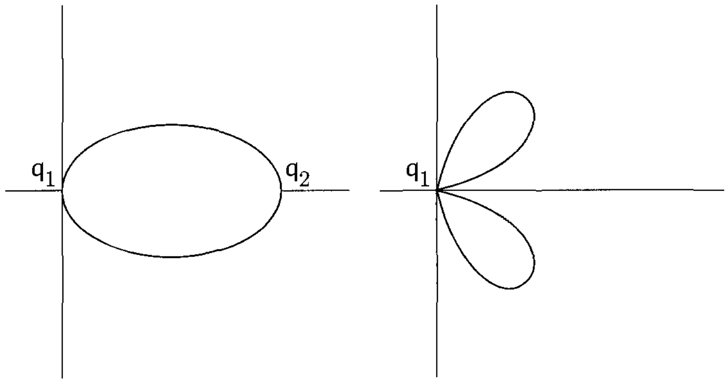

Figure 1

from the q_1 and ending in a different point q_2, let $d(q_1, q_2) = d < \infty$ be the distance between these two points. Slit M along α. As a next step we identify q_1 and q_2 obtaining a surface homeomorphic to a new flat surface minus two open disks (fig. 1), both with perimeter d. Finally, we glue both boundaries of the disks to the boundary circles of a finite flat cylinder $S^1_d \times [0, 1]$ (S^1_d denotes a circumference having perimeter d) to form a handle. In the new singular flat structure $q_1 = q_2$ is a zero of order $k_m + 4$.

Construction 2. If $(k_1, \ldots, k_m;\ \epsilon) \in \mathcal{K}$ is realized by a singular flat structure (M, ϕ), then $(k_1, \ldots, k_m, 4;\ \epsilon)$ is also realizable.

To produce an isolated zero of order 4, take a segment of geodesic $\alpha : [0, d] \to M$, with $\alpha\,(0) = q_1$, $\alpha\,(d) = q_2$, and without singular points along it. Cut along α and identify q_1 and q_2 to form a surface with two boundary circles of perimeter d, joined at $q_1 = q_2$. Attach a finite flat cylinder $S^1_d \times [0, 1]$, to get a handle. In the new singular flat structure, p is a zero of order 4.

Construction 3. If $(k_1, \ldots, k_m;\ \epsilon) \in \mathcal{K}$ is realized by a singular flat structure (M, ϕ), then $(k_1, \ldots, k_m, 2, 2;\ \epsilon)$ is also realizable.

To produce two zeros of order 2, slit M along a nonsingular geodesic segment $\alpha : [0, d] \to M$, $\alpha\,(0) = q_1$, $\alpha\,(d) = q_2$, obtaining a new surface homomorphic to M minus an open disk. Slit also a nonsingular flat torus T along a geodesic segment $\beta : [0, d] \to T$, $\beta\,(0) = p_1$, $\beta\,(d) = p_2$, of the same length $d(p_1, p_2) = d$. Glue each side of the cut in M adequately to each side of the

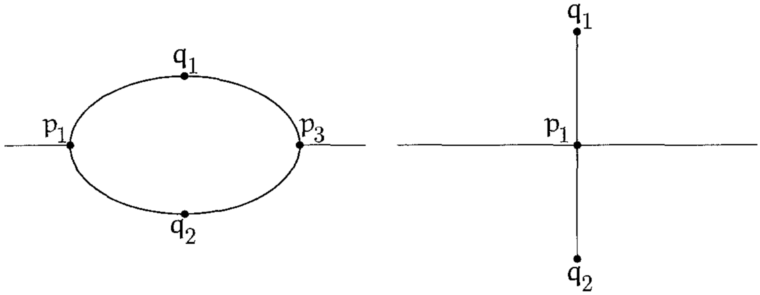

Figure 2

cut in T, identifying p_1 with q_1, and also p_2 with q_2. In the new singular flat structure. $p_1 = q_1$ and $p_2 = q_2$ are two new zeros whose order is 2.

Construction 4. If $(k_1, \ldots, k_m;\ \epsilon) \in \mathcal{K}$ is realized by a singular flat structure (M, ϕ), then $(k_1, \ldots, k_m, 1, 1, 2;\ -1)$ is also realizable.

Slit M along two adjacent nonsingular geodesic segments $\alpha, \beta : [0, d] \to M$, suppose that $\alpha(0) = p_1$, $\alpha(d) = p_2$, and $\beta(0) = p_2, \beta(d) = p_3$. Attach a handle to the remaining boundary of two circumferences that touch each other in p_2. It turns out that, p_2 has become a zero having order two and p_1 and p_3 have become simple zeros.

Construction 5. If $(k_1, \ldots, k_m;\ \epsilon) \in \mathcal{K}$ is realized by a singular flat structure (M, ϕ), then $(k_1, \ldots, k_m, 1, 1, 1, 1;\ -1)$ is also realizable.

Slit M along two not intersecting nonsingular geodesic segments of the same length. Suppose that these segments have extreme points p_1, p_2 and q_1, q_2; attach a suitable handle to both cuts, obtaining four simple zeros at these four points.

Construction 6. If $(k_1, \ldots, k_m;\ \epsilon) \in \mathcal{K}$ is realized by a singular flat structure (M, ϕ), then $(k_1, \ldots, k_m, -1, -1, 2;\ -1)$ is also realizable.

Slit M along a nonsingular geodesic segment $\alpha : [0, 2d] \to M$, $\alpha(0) = p_1$, $\alpha(d) = p_2$, $\alpha(2d) = p_3$. On each side of the cut, take the midpoints q_1, q_2 (note that q_1, q_2 are both identical to p_2 if we do not slice M). Glue identifying $\overline{p_1 q_1}$ with $\overline{p_3 q_1}$ and $\overline{p_1 q_2}$ with $\overline{p_3 q_2}$. After we have made the identifications, $p_1 = p_3$ become a zero having order 2, q_1 and q_2 become simple poles (fig. 2).

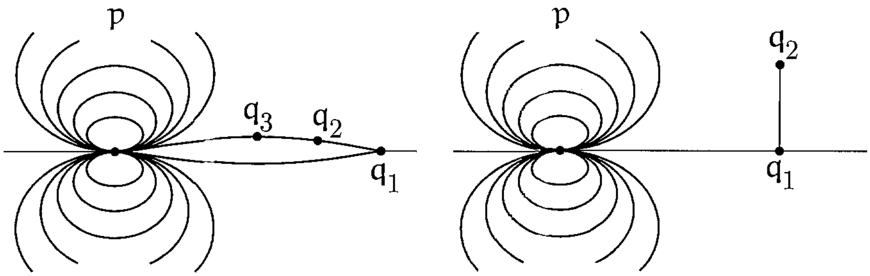

Figure 3

Construction 7. If $\kappa = (k_1, \ldots, k_m;\ \epsilon) \in \mathcal{K}$ is realized by a singular flat structure (M, ϕ), then $(k_1, \ldots, k_m, -1, -1, -1, -1,\ 4; -1)$ is also realizable.

Cut along $\alpha : [0, 4d] \to M$, $\alpha(0) = p_0$, $\alpha(4d) = p_4$. In each of both segments that we have obtained and whose extremes are p_0 and p_4, take three points p_1, p_2, p_3 in one of them and three points q_1, q_2, q_3 in the other one in such a way that each one of the original segments is now divided in four segments of the same length d. Then do the following gluing operations or identifications: $\overline{p_0p_1}$ to $\overline{p_1p_2}$ identifying p_0 with p_2, $\overline{p_2p_3}$ to $\overline{p_3p_4}$ identifying p_2 with p_4, $\overline{p_0q_1}$ to $\overline{q_1q_2}$ identifying p_0 with q_2, $\overline{q_2q_3}$ to $\overline{q_3p_4}$ identifying q_2 with p_4. Thus we have simple poles at p_1, p_3, q_1, q_3 and a zero of order 4 at $p_0 = p_2 = q_2 = p_4$.

Construction 8. If $\kappa = (k_1, \ldots, k_m;\ \epsilon) \in \mathcal{K}$ is realized by a singular flat structure (M, ϕ), then $(k_1, \ldots, k_m, -1,\ 1; -1)$ is also realizable.

Let us consider the following cases:

Case 1. There is an entry $k_i \leq -3$. This means that there is a pole p of order at least 3, i.e., if α is a geodesic critical trajectory joining p to different regular point q_1, $d(p, q_1) = \infty$, where $d(p, q_1)$ is the length measured along the geodesic α, $\alpha : [0, \infty) \to M$, $\alpha(0) = q_1$, $\lim_{t\to\infty} \alpha(t) = p$. Cut M along α from p to q_1. Take two consecutive points q_2 and q_3 on one side of the cut in such a way that both segments $\overline{q_1q_2}$ and $\overline{q_2q_3}$, have the same length. Glue isometrically this two segments identifying q_1 with q_3, after this we get a simple pole of order -1 in q_2. As a second step glue isometrically the two sides of the infinite cut which goes from $q_1 = q_3$ to p. It turns out that, q_1 becomes a simple zero, and p remains as a pole having the same order as before (fig. 3).

Case 2. There is an entry $k_i = -2$. Let p be a pole having order 2, locally we have trivial holonomy so we can think it as a sink, a source or a periodic center of a vector field. In the first two cases we have infinite geodesics that reach the extreme of the cylinder in an infinite arc-length; in the last situation we have finite closed geodesics. In both cases we have the geometry of an infinite cylinder $S^1_{3d} \times [0, \infty)$, the singularity corresponds to the infinite end of this cylinder. Cut this cylinder along a circumference that is a closed geodesic (in the first case this geodesic is one of the geodesics given as usual by our horizontal foliation, in the second case this geodesic is transverse to the infinite geodesics that reach the pole), suppose that this circumference has perimeter $3d$, take three points p_1, p_2, p_3 cyclically in it such that

$$d(p_i, p_j) = d \quad \text{for} \quad i \neq j.$$

Glue the segments $\overline{p_1 p_2}$ and $\overline{p_2 p_3}$ identifying p_1 to p_3, p_2 becomes a pole of order one and the resulting surface has a circumference of perimeter d as its boundary. Glue a cylinder $S^1_d \times [0, \infty)$ to this circumference (the foliation can be conveniently extended). Thus $p_1 = p_3$ becomes a zero having order 1 and we have again a pole having order two at the infinite end of the cylinder.

Now we are ready to prove the following

Lemma 1. *Suppose $\kappa, k \in \mathcal{K}$. If $\kappa \precsim k$ and κ is realizable, then k is also realizable.*

Proof. Let $k = (k_1, \dots, k_n;\ \epsilon)$ be in $\mathcal{K}$, suppose that $\kappa = (k_1, \dots, k_m;\ \epsilon)$ is realizable and such that $\kappa \precsim k$ (assume the notation as in the definition of the partial ordering). Let r_i be the number of entries k_j of k ($j = l+1, \dots, n$) such that $k_j \equiv i \pmod 4$, for $i = -1, 1, 2, 4$, and let s_i be the number of entries k_j of κ ($j = l+1, \dots, m$) with the same property.

Construction 1 implies that we have to consider only the problem of obtaining $r_i - s_i$ singularities of order i, for $i = -1, 1, 2, 4$. Furthermore, by using construction 2 we can have $r_4 - s_4$ zeros of order 4. Therefore we only need to get $r_i - s_i$ singularities of order i, for $i = -1, 1, 2$.

We claim that constructions 3 to 8 are enough to have the required number of singularities of order $-1, 1, 2$. The claim is true since with the operations described in the given constructions we are able to construct any number of singularities of these orders.

We summarize the whole process in the following diagram.

Applying constructions	$\varkappa \in \mathcal{K}$	$\sigma(\varkappa)$
	$(k_1, \dots, k_m;\ \epsilon)$	$4g'' - 4$
6, 7, 8	$\downarrow$	
	$(k_1, \dots, k_m, -1, \dots, 1, \dots, 2, \dots, 4, \dots;\ \epsilon)$	$4g'' - 4$
2, 3, 4, 5	$\downarrow$	
	$(k_1, \dots, k_m, -1, \dots, 1, \dots, 2, \dots, 4, \dots;\ \epsilon)$	$4g' - 4$
1	$\downarrow$	
	$(k_1, \dots, k_m, -1 + 4h_1, \dots, \dots, 4 + 4h_{n-m};\ \epsilon)$	$4g - 4$

$$0 \leq h_i \in \mathbb{Z},\ g'' < g' < g.$$ □

4.3 Realization of minimal data

Let us give a more precise characterization of the minimal elements just by an arithmetical observation.

Proposition. *If $\kappa \in \mathcal{K}$ is a minimal element then $\sigma(\kappa) \in \{-4, 0\}$.*

Proof. Take $\kappa = (k_1, \dots, k_m;\ \epsilon)$ and that $\sigma(\kappa) \geq 4$, we will show that κ is by no means minimal.

If $k_m \geq 3$, then we claim that $(k_1, \dots, k_m - 4;\ \epsilon)$ satisfies all conditions (1) to (5) of our main Theorem and therefore belongs to $\mathcal{K}$. Furthermore

$$(k_1, \dots, k_m - 4;\ \epsilon) \precsim \kappa$$

and therefore κ is not minimal.

So let us take

$$\kappa \doteq (k_1, \dots, k_l, -1, \dots, -1, 1, \dots, 1, 2, \dots, 2;\ \epsilon)$$

where $k_i \leq -2$ for $i = 1, \dots, l$. Define $n_{-1} \geq 0$ as the number of entries of κ that are equal to -1, define n_1 and n_2 in the same way. Then, our first assumption states that

$$-n_{-1} + n_1 + 2n_2 + \sum_{i=1}^{l} k_i = \sigma(\kappa) \geq 4$$

hence

$$2n_2 + n_1 \geq 4$$

which implies $n_1 \geq 2$ or $n_2 \geq 2$.

If $n_1 \geq 2$ and $n_2 \geq 1$ then $\kappa = (k_1, \dots, k_l, \dots k_{m-3}, 1, 1, 2;\ \epsilon)$. Then we can check that $(k_1, \dots, k_l, \dots k_{m-3};\ \epsilon) \in \mathcal{K}$ and

$$(k_1, \dots, k_l, \dots k_{m-3};\ \epsilon) \precsim \kappa.$$

If $n_1 \geq 2$ and $n_2 = 0$, then $n_1 \geq 4$ and therefore

$$\kappa = (k_1, \dots, k_l, \dots k_{m-4}, 1, 1, 1, 1;\ \epsilon)$$

then $(k_1, \dots, k_l, \dots k_{m-4};\ \epsilon) \in \mathcal{K}$ and

$$(k_1, \dots, k_l, \dots k_{m-4};\ \epsilon) \precsim \kappa.$$

If $n_2 \geq 2$, then $\kappa = (k_1, \dots, k_l, \dots, k_{m-2}, 2, 2;\ \epsilon)$, and again we have a new element $(k_1, \dots, k_l, \dots k_{m-2};\ \epsilon) \in \mathcal{K}$

$$(k_1, \dots, k_l, \dots k_{m-2};\ \epsilon) \precsim \kappa.$$

In any case, κ is not minimal. □

Lemma 2. *Every minimal element $\kappa = (k_1, \dots, k_m; \epsilon) \in \mathcal{K}$ is realizable.*

Proof. Since κ is minimal $\sigma(\kappa) = -4, 0$.

Case 1. $\sigma(\kappa) = -4$. We have the realization problem on the Riemann sphere, and we have already solved it.

Case 2. $\sigma(\kappa) = 0$. If $\epsilon = +1$, then we conclude that the only minimal case that occurs on the torus is $(-2, 2; +1)$ but this element (and all that are greater than this) presents an obstruction by the Residue Theorem, therefore it does not belong to $\mathcal{K}$. Hence $\epsilon = -1$.

We want to construct all the singular flat structures on the torus, considering structures on the sphere. The difficulty in this approach arises when, in certain cases, we want to change the orientability of the horizontal foliation. In order to solve this problem let us consider the following additional basic constructions.

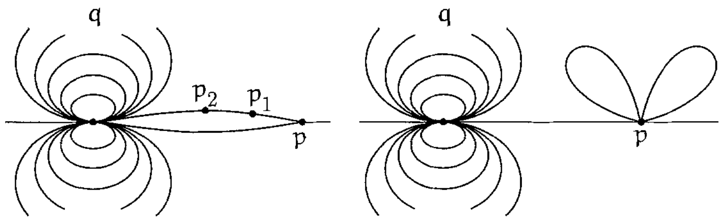

Figure 4

Construction 9. If a singular flat structure on the sphere $(\mathbb{CP}^1, \phi)$ realizes $(k_1, \ldots, k_m; \epsilon)$ where $k_1 \leq -2$, then

$$(k_1, \ldots, k_m, 4; -1)$$

is also realizable (on the torus).

Let $q \in \mathbb{CP}^1$ be a pole of order $-k_1$, and let $\alpha : [0, \infty) \to \mathbb{CP}^1$ be a geodesic of the horizontal foliation such that $\alpha(0) = p$ is a regular point, and $\lim_{t\to\infty} \alpha(t) = q$. Cut along α, on one side of the cut take two consecutive points p_1, p_2 such that both segments $\overline{pp_1}$, $\overline{p_1p_2}$ have the same length d. Identify p, p_1 and p_2. Glue the other side of the cut to de infinite lengthened segment that goes from the previous identification point to q (fig. 4). Attach a cylinder $S^1_d \times [0, 1]$ to the remaining boundary components tangent at p. We have thus obtained a zero of order 4 at $p = p_1 = p_2$. It can be checked that the resulting singular flat structure has non orientable horizontal foliation.

Construction 10. Suppose that singular flat structure on the sphere $(\mathbb{CP}^1, \phi)$ realizes $(k_1, \ldots, k_m; \epsilon)$, ϕ having a pole q of order $-k_1 \geq 2$, a zero of order $k_m \geq 0$ and a geodesic of the horizontal foliation connects p and q. Then

$$(k_1, \ldots, k_{m-1}, k_m + 4; -1)$$

is also realizable (on the torus).

Let $\alpha : [0, \infty) \to \mathbb{CP}^1$ be a geodesic of the horizontal foliation such that $\alpha(0) = p$ and $\lim_{t\to\infty} \alpha(t) = q$. Cut along α and proceed as in the previous construction in order to get a zero of order $k_m + 4$ at p.

Construction 11. If a singular flat structure on the sphere $(\mathbb{CP}^1, \phi)$ realizes $(k_1, \ldots, k_m; \epsilon)$ where $k_1 \leq -2$, $k_m \geq 0$, then

$$(k_1, \ldots, k_{m-1}, k_m + 4; -1)$$

is also realizable (on the torus).

From the previous construction we see that the only thing we need to achieve this construction, is to give a particular singular flat structure on the sphere such that there exists a pole q of order $-k_1$ connected to a zero p of order k_m, by means of a geodesic of the horizontal foliation. Suppose that $(k_1, \ldots, k_m; \epsilon) = (k_1, \ldots, k_l, \ldots, k_m; \epsilon)$ where $k_i \in \mathbb{Z}^-$ for $i = 1, \ldots, l$ and $k_j \in \mathbb{Z}^+$ for $i = l+1, \ldots, m$. Take $a_2 < a_3 < \cdots < a_m < a_1$ real numbers in $\mathbb{C}$, an take the following quadratic differential (given in the affine chart)

$$\frac{(z-a_{l+1})^{k_{l+1}} \cdots (z-a_m)^{k_m}}{(z-a_1)^{k_1} \cdots (z-a_l)^{k_l}} dz^2.$$

It can be easily verified that this meromorphic quadratic differential on the sphere satisfies the hypothesis of construction 10.

Construction 12. If $(k_1, \ldots, k_m, -2;\ \epsilon)$ is realized by a singular flat structure (M, ϕ), then

$$(k_1, \ldots, k_m, 2;\ -1)$$

is also realizable.

Cut the cylinder $S^1_{2d} \times [0, \infty)$ which produces the order two pole along a closed geodesic of length $2d$ (not necessarily of the horizontal foliation). Take two points p an q on the boundary we have just obtained, in such a way that they divide it into two segments of the same length d. Identify p with q and attach a handle where necessary. Thus instead of the pole of order 2 we get a zero having order 2.

Now that we have developed these tools (constructions 9 to 12), let us continue now with the proof of lemma 2. We remind the reader that our purpose is to solve the realization problem on the torus under the hypothesis that the prescribed that the horizontal foliation is non orientable. Since $\kappa = (k_1, \ldots, k_{m-1}, k_m; -1) \in \mathcal{K}$ and $\sigma(\kappa) = 0$, we can assume that $k_1 \leq -2$ and $k_m \geq 1$. Consider $\left(k_1, \ldots, k_{m-1}, k'_m; \epsilon'\right)$ where $k'_m := k_m - 4$, and $\epsilon' = +1$ if k_m and k_i are even for each i or $\epsilon' = -1$ otherwise. Observe that we have defined topological data that belong to $\mathcal{K}$. Furthermore, this new element is realizable on the sphere.

If $k_m \geq 4$, then $k'_m \geq 0$ and by applying constructions 9 and 11 we have a singular flat structure that realizes κ.

If $k_m = 3$, then $k'_m = -1$, $\epsilon' = -1$ and therefore by applying construction 2 (which does not change the orientability status of the foliation), we have that κ is realizable. Moreover, in this case κ is not minimal.

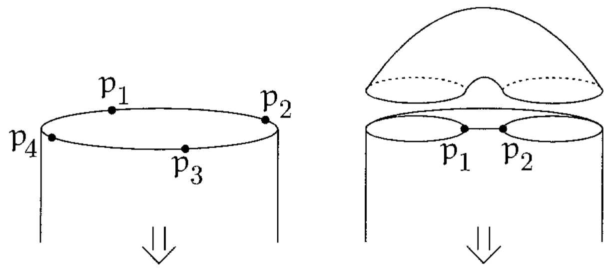

Figure 5

If $k_m = 2$, then according to construction 12 κ is realizable.

If $k_m = 1$, we can assume that every entry of κ is less or equal to 1. Let n_1 the number of entries of κ that equal 1.

If $n_1 \geq 4$, then we can write $\kappa = (k_1, \ldots, k_m, 1, 1, 1, 1; -1)$, take a new element $(k_1, \ldots, k_m; \epsilon') \in \mathcal{K}$ realizable on the sphere (ϵ' defined as above). Then apply construction 5 to conclude that κ is realizable.

If $n_1 = 1$ we can not have an entry less or equal to -2. Therefore this case is not tenable.

If $n_1 = 2$ or 3, then by the minimallity of κ, we have few cases to consider, namely $(-2, 1, 1; -1)$, $(-3, 1, 1, 1; -1)$ both are realizable as it is shown below.

$(-2, 1, 1; \ -1)$: Consider a flat cylinder $S^1_{4d} \times [0, \infty)$, and take $p_1, \ p_2, \ p_3$ and p_4 in the boundary circle in such a way that we form segments of the same length $|\overline{p_i p_{i+1}}| = |\overline{p_4 p_1}| = d$, $i = 1, 2, 3$, glue $\overline{p_1 p_2}$ to $\overline{p_3 p_4}$, identifying p_1 to p_4 and p_2 to p_3. Attach a suitable handle in the boundary circles that still remain. Thus we have formed two zeros of order one at $p_1 = p_4$ and $p_2 = p_3$, we also have one pole of order 2 (fig. 5).

$(-3, 1, 1, 1; \ -1)$: Take the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ and take the quadratic differential $\phi \equiv 1$ near 0 in the affine chart of $\mathbb{C}$. In order to get an admissible parameter for this differential in a domain that contains ∞, apply the inversion $z \longmapsto \tilde{z} = \frac{1}{z}$; then we can see that the transformation rule yields

$$\tilde{\phi}(\tilde{z}) = \phi(z) \left(\frac{dz}{d\tilde{z}} \right)^2 = \frac{1}{\tilde{z}^4}.$$

Thus we have a pole of order 4 at ∞, and the geometry of the foliation is given by taking the inverse image, under the stereographic projection, of the usual horizontal foliation in $\mathbb{C}$. Any horizontal trajectory is critical has infinite length with both rays tending to ∞. Cut $\mathbb{CP}^1$ along one of these trajectories α : $(-\infty, \infty) \to \mathbb{CP}^1$, say along the one that corresponds to $\{(x, 0) \mid \ x \in \mathbb{R}\} \subset \mathbb{C}$

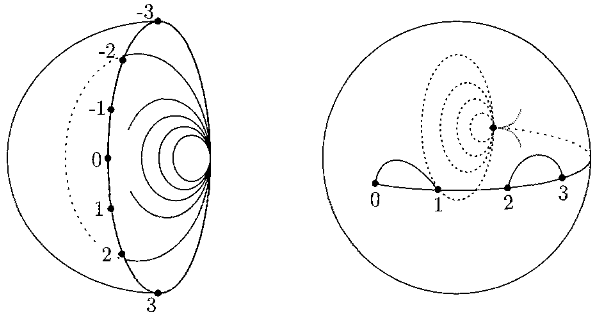

Figure 6

under stereographic projection; assume $\alpha(0) = 0 \in \mathbb{CP}^1$. After we cut we have two hemispheres. Take one of these hemispheres, we can parametrize its geodesic boundary with α as above. Identify $\alpha(t)$ with $\alpha(-t)$ along the boundary, for $t \in [1, 2] \cup [3, \infty)$. Note that the border of the new surface has two components of the same length (fig. 6), attach a suitable cylinder to them. Then we have a regular point at $\alpha(0)$ and three zeros of order 1 at $\alpha(1)$, $\alpha(2)$ and $\alpha(3)$, and a pole of order 3 at the point that corresponds to $\lim_{t\to\infty} \alpha(t)$.

We showed that in any possible case κ is realizable.

A more concise approach is given by the following diagram.

Applying constructions	$\varkappa \in \mathcal{K}$	$\sigma(\varkappa)$
	$(k_1, \ldots, k_{m-1}, k_m;\ \epsilon')$	-4
11, 9, 5	$11 \swarrow \quad 5 \downarrow \quad \searrow 9$	
resp.	$(k_1, \ldots, k_m + 4; -1) \downarrow (k_1, \ldots, k_m, 4; -1)$	0
	$(k_1, \ldots, k_m, 1, 1, 1, 1;\ -1)$	0
	$(k_1, \ldots, k_m, -1;\ -1)$	-4
2	$\downarrow$	
	$(k_1, \ldots, k_m, 3;\ -1)$	
	(Not minimal)	0
	$(k_1, \ldots, k_m, -2;\ \epsilon')$	-4
12	$\downarrow$	
	$(k_1, \ldots, k_m, 2;\ -1)$	0
By hand	$(-3, 1, 1, 1; -1)$, $(-2, 1, 1; -1)$	0

□

From lemmas 1 and 2 we conclude the following

Corollary 1. *If $k \in \mathcal{K}$, then k is realizable.* □

As we observed, Theorem 1 follows from this corollary and from the main theorem in **[MS]**. Therefore, the proof is complete.

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Homero G. Diaz-Marin
Instituto de Matemáticas UNAM, Unidad Morelia
Apartado Postal 61-3 Xangari C.P. 58089
and
Escuela de Ciencias Físico Matemáticas,
UMSNH. Edif. B, C.U.
Morelia, Michoacán,
MEXICO.

E-mail: homero@matmor.unam.mx